A family of cuts for the matroidal knapsack problem

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Abstract

A family of cuts is given for the polyhedron of the intersection of a matroid with a knapsack problem. These cuts could be combined with matroid and knapsack cuts in a Branch and Cut Method, for this we give some heuristics to obtain such cuts. Some of the cuts give facets of small problems that can be lifted to facets of large problems.

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1 Problem statement and definitions

We shall be concerned about Independent Systems, one corresponding to the *Knapsack Problem*, where we assume that each element j of E is given a non-negative integer weight a_j and that the family of independent sets, denoted by $\mathcal{K}$, is defined as the family of subsets I of E such that $\sum_{j\in I} a_j \leq b$, where b is a given positive integer.

The other special Independent System we shall be considering is obtained when the following property holds for $\mathcal{I}$: for all $I, J \in \mathcal{I}$ such that $|I| = |J|+1$, there exists an element k of $I \setminus J$ such that $I \cup \{k\} \in \mathcal{I}$. In this case the Independent System is called a *Matroïd* and its family of independent sets is denoted by $\mathcal{M}$. The reader is referred to [5] for Matroid Theory fundamentals and notation.

This work studies properties of the Independent System resulting of the intersection of matroid and knapsack Independent System defined over the same ground set E. The simplest non-trivial such intersections is obtained when the family of independent sets of an Independent Systems must obey both a knapsack and a matroid constrains. We then speak of *Matroid Knapsack* problems and we denote by $\mathcal{L}$ the family of its independent sets. These problems have been considered in [2] from the point of view of lagrangean relaxation and its properties. The case of several knapsack constraints has been studied in [1]. Here we begin to analyze the polyhedral structure of these problems.

The polyhedral approach identifies an Independent System using the characteristic vector x^I given for each I by $x^I_j = 1$ if j in I, 0 otherwise, and considering the convex hull of these vectors, i. e. $P_{\mathcal{I}} := conv\{x^I \in \{0,1\}^n : I \in \mathcal{I}\}$.

We denote by $\mathcal{D}$ the family of all subsets of E which are not independent, called the family of *dependent sets* $\mathcal{D} = \{D \in E : D \notin \mathcal{I}\}$, and by $\mathcal{C}$ the family of all minimal dependent sets or *circuits* of the Independent System.

For any circuit C of Independent System a valid inequality for the corresponding polyhedron $P_{\mathcal{I}}$ is given by $\sum_{j \in C} x_j \leq |C| - 1$

In order to simplify the notation in the inequalities, we use $x(A)$ as a short-hand for $\sum_{j \in A} x_j$, for any subset A of E.

More generally, for any $A \subseteq E$, a valid inequality for $P_{\mathcal{I}}$ is given by

$$x(A) \leq r(A) \tag{1}$$

where $r(A) = \max\{|I| : I \in \mathcal{I} \text{ and } I \subseteq A\}$, is called the rank function of the Independent System.

It is well known that $P_{\mathcal{M}}$ has been fully characterized in [3] as the set of real vectors x satisfying the inequalities (1) and non-negativity. As for as $P_{\mathcal{K}}$ is concerned, no full characterization is known, but this polyhedron has been extensively studied and several families of facet-inducing inequalities have been found. The reader is referred to [4] as an up-to-date text in integer programming and combinatorial optimization.

2 Valid inequalities for Matroidal Knapsack problems

We will derive cuts for $P_{\mathcal{L}}$ from $P_{\mathcal{K}} \cap P_{\mathcal{M}}$.

2.1 Trough the paper, we denote by (P, Q, S, T) a partition of E, with cardinalities p, q, s, t respectively, by k, the rank of $(P \cup T)$ in the matroid $(E, \mathcal{M})$ and by q_0 a distinguish element of Q.

Assumption 2.2 We assume: (a) $t \geq 2, q \geq 1, 0 \leq p < k < p+t$. (b) $r_{\mathcal{M}}(Q \cup P \cup T) = k+q$. (c) The set $P \cup T$, which is dependent in the matroid $(E, \mathcal{M})$, is independent in the knapsack $(E, \mathcal{K})$. (d) For all $e \in T$ the set $P \cup Q \cup \{e\}$ is a cycle of the knapsack $(E, \mathcal{K})$ and is independent in the matroid $(E, \mathcal{M})$. (e) The set S, which represent the elements not used in the configuration, could be empty.

We do not know a complete description for $P_{\mathcal{K}}$, hence we do not expect to find a complete description for $P_{\mathcal{L}}$. But it has to satisfy the rank inequalities (1) for both $P_{\mathcal{K}}$ and $P_{\mathcal{M}}$ and any non-negative linear combination of them. In the next lemma we identify some valid inequalities for $P_{\mathcal{L}}$ corresponding to Assumption 2.2 that we use to derive the cuts.

Lemma 2.3 Apart from $0 \leq x_e \leq 1 \; \forall e \in E$, the following inequalities are valid for $P_{\mathcal{L}}$.

$$x(P \cup T) \leq k \tag{2}$$

$$x(P \cup Q \cup \{e\}) \leq p + q \;\forall e \in T \tag{3}$$

$$2x(T) + (t+1)x(P) + tx(Q) \leq k + t(p+q) \tag{4}$$

$$2x(T) + (t+1)x(P \cup Q) \leq k + t(p+q) + q \tag{5}$$

$$2x(T) + (t+1)x(P \cup Q \setminus \{q_0\}) + tx_{q_0} \leq k + t(p+q) + q - 1 \tag{6}$$

$$2x(T) + (t+2)x(P) + tx(Q) \leq k + t(p+q) + p \tag{7}$$

Proof: The inequalities (2) and (3) are a direct consequence of inequality (1) and Assumption 2.2. Adding the inequality in (2) and the inequalities in (3) we obtain (4). Adding $x_e \leq 1$ for all $e \in Q$ to (4) we obtain (5). Adding $x_e \leq 1$ for all $e \in Q \setminus \{q_0\}$ to (4) we obtain (6). Adding $x_e \leq 1$ for all $e \in P$ to (4) we obtain (7). □

In the next lemma we define x^o, a fractional vertex of $P_{\mathcal{M}} \cap P_{\mathcal{K}}$ which is cut by the inequalities of Theorem 2.5.

Lemma 2.4 Let x^o be defined as: $x^o_e = 1$ if $e \in P \cup Q \setminus \{q_0\}$; $x^o_e = \frac{k-p}{t}$ if $e \in T$; $x^o_e = 1 - \frac{k-p}{t}$ if $e = q_0$; $x^o_e = 0$ if $e \in S$.

Then x^o is a *fractional vertex* of $P_{\mathcal{M}} \cap P_{\mathcal{K}}$.

Proof:

1) Clearly $\vec{0} \leq x^o \leq \vec{1}$ and $0 < x^o_{q_0} < 1$ since $p < k < p+t$ by Assumption 2.2, hence it is fractional. Therefore we could write:

$$x^o = x^{P \cup Q \setminus \{q_0\}} + (1 - \tfrac{k-p}{t})x^{\{q_0\}} + (\tfrac{k-p}{t})x^T \quad (a)$$

$$\leq x^{P \cup Q \setminus \{q_0\}} + (1 - \tfrac{k-p}{t})x^{\{q_0\}} + (\tfrac{t-1}{t})x^T \quad (b)$$

2) Since (a) is equivalent to:

$$(1 - \frac{k-p}{t})x^{P \cup Q} + (\frac{k-p}{t})x^{P \cup T \cup Q \setminus \{q_0\}}$$

We have $x^o \in P_{\mathcal{K}}$, because x^o is a convex combination of two points in $P_{\mathcal{K}}$.

3) Since (b) is equivalent to:

$$(1 - \tfrac{k-p}{t})x^{P \cup Q} + (\tfrac{t-1}{t})(1 - \tfrac{k-p}{t})x^T + (\tfrac{k-p}{t})x^{P \cup Q \setminus \{q_0\}} + (t-1)(\tfrac{k-p}{t^2})x^T$$

$$= \textstyle\sum_{e \in T} \frac{1}{t}(1 - \frac{k-p}{t})x^{P \cup Q \cup T \setminus \{e\}} + \sum_{e \in T} \frac{k-p}{t^2} x^{P \cup Q \setminus \{q_0\} \cup T \setminus \{e\}}$$

We have $x^o \in P_{\mathcal{M}}$, because x^o is lest than or equal to a convex combination of points in $P_{\mathcal{M}}$.

4) $x^o(P \cup T) = x^o(P) + x^o(T) = p + (k-p) = k$

$x^o(P \cup Q \cup \{e\}) = p + (q-1) + (1 - \frac{k-p}{t}) + (\frac{k-p}{t}) = p + q \;\forall e \in T$

Hence the equation system of x^o contains:

$$\begin{array}{lll} \text{i)} & x(P \cup T) = k & \\ \text{ii)} & x(P \cup Q \cup \{e\}) = p + q & \forall e \in T \\ \text{iii)} & x_e = 1 & \forall e \in P \cup Q \setminus \{q_0\} \\ \text{iv)} & x_e = 0 & \forall e \in S \end{array}$$

The only solution to system (i,ii,iii,iv) is x^o since deducting $x_e = 1 \; \forall e \in P$ from (i), and $x_e \overset{o}{=} 1 \; \forall e \in Q \setminus \{q_0\}$ from (iii) we obtain:

i') $x(T) = k - p$

ii') $x_e + x_{q_0} = 1 \; \forall e \in T$

Therefore adding (ii') for all $e \in T$ and using (i'), we have

$x(T) = t(1 - x_{q_0}) = k - p$.

That is $x_{q_0} = 1 - \frac{k-p}{t}$ and $x_e = \frac{k-p}{t} \; \forall e \in T$. Hence x^o is a basic solution.

□

The main result of this paper is the next theorem.

Theorem 2.5 The following inequalities are cuts for $P_{\mathcal{L}}$.

$$x(T) + tx(P) + (t-1)x(Q) \leq t(p+q) - q \tag{8}$$

$$x(T \cup Q) + 2x(P) \leq k + p + q - 1 \tag{9}$$

When t and $k + t(p+q) + q$ are odd:

$$x(T) + \frac{t+1}{2}x(P \cup Q) \leq \frac{k + t(p+q) + q - 1}{2} \tag{10}$$

When t is odd and $k + t(p+q) + q$ is even:

$$x(T) + \frac{t+1}{2}x(P \cup Q \setminus \{q_0\}) + \frac{t-1}{2}x_{q_0} \leq \frac{k + t(p+q) + q - 2}{2} \tag{11}$$

When t is even and $k + t(p+q) + p$ is odd:

$$x(T) + \frac{t+2}{2}x(P) + \frac{t}{2}x(Q) \leq \frac{k+t(p+q)+p-1}{2} \qquad (12)$$

When t and $k + t(p+q) + p$ are even:

$$x(T) + \frac{t+2}{2} + x(P) + \frac{t}{2}x(Q) \leq \frac{k+t(p+q)+p-2}{2} \qquad (13)$$

Proof : The proofs of cuts (8) and (9) are similar. We prove only (8).

Replacing x^o in (8) we have:

$x^o(T) + tx^o(P) + (t-1)x^o(Q) =$

$(k-p) + tp + (t-1)(q-1) + (t-1)(1 - \frac{k-p}{t}) =$

$(k + t(p+q) - q) + \frac{k-p}{t} > k + t(p+q) - q$

To show that it is a valid inequality for $P_{\mathcal{L}}$, consider any set $A \in \mathcal{L}$.

If $P \not\subseteq A$ then $x^A(P) \leq p-1$ and the vector x^A satisfy (8) since

$x^A(T) + tx^A(P) + (t-1)x^A(Q) \leq t + t(p-1) + (t-1)q = t(p+q) - q$.

If $P \subseteq A$ then:

either $Q \subseteq A$ which implies that not element of T is in the solution, in this case the vector x^A satisfy (8) since

$x^A(T) + tx^A(P) + (t-1)x^A(Q) = 0 + tp + (t-1)q = t(p+q) - q$.

or $x^A(Q) \leq q-1$ and $x^A(T) \leq k-p \leq t-1$, because $k = r(P \cap T)$ and $k < t+p$, in this case the vector x^A satisfy (8) since

$x^A(T) + tx^A(P) + (t-1)x^A(Q) \leq t - 1 + tp + (t-1)(q-1) = t(p+q) - q$.

The proofs of cuts (10), (11) and (12) are similar, but with a different technique from the proof of (8) and (9). We prove only (10).

If t and $k + t(p+q) + q$ are odd then the inequality (10) is a valid inequality for $P_{\mathcal{L}}$ which cuts x^o.

Dividing by two the valid inequality (5)

$2x(T) + (t+1)x(P \cup Q) \leq k + t(p+q) + q$

and taking the floor of it right-hand-side gives (10).

Replacing x^o in twice (10) we have:

$2x^o(T) + (t+1)x^o(P \cup Q) =$

$2(k-p) + (t+1)(p+q-1) + (t+1)(1 - \frac{k-p}{t}) =$

$2k - 2p + tp + tq - t + p + q - 1 + t + 1 - k + p - \frac{k-p}{t} =$

$k + (-2 + t + 1 + 1)p + (t+1)q - \frac{k-p}{t} =$

$(k + t(p+q) + q - 1) + (1 - \frac{k-p}{t}) > k + t(p+q) + q - 1.$

The proof of cut (13) uses cuts (8) and (9). □

3 Facets for $P_{\mathcal{L}}$

In this section we give an example that shows how to derive conditions which assure us that some of the inequalities in Theorem 2.5 correspond to facets of $P^*_{\mathcal{L}} = P_{\mathcal{L}} \cap \{x_e = 0 : e \in S\}$.

Hence, we can use standard Lifting Techniques to obtain facets of $P_{\mathcal{L}}$.

To show that a given valid inequality of a polyhedron P is facet inducing it is enough to show that that there are dimension of P points in P which are linear independent and satisfy the given inequality as equality. Notice that dimension of $P^*_{\mathcal{L}}$ is $p+q+t$.

Example 3.1 Let $E = \{1, \ldots, t, p_0, q_0\}$, $T = \{1, \ldots, t\}$, $P = \{p_0\}$ and $Q = \{q_0\}$, with $p = q = 1$ and $k = t$ and let $T \cup Q \in \mathcal{K}$. In this case the dimension of $P^*_{\mathcal{L}}$ is $t+2$.

In $P^*_{\mathcal{L}}$, the $t+2$ points $x^{P \cup T \setminus \{e\}}$, $\forall e \in T$, $x^{P \cup Q}$, $x^{T \cup Q}$ are independent since it is easy to solve. The inequality (8) is facet inducing for $P^*_{\mathcal{L}}$ since in this case this inequality is: $x(T) + tx(P) + (t-1)x(Q) \leq 2t - 1$ which is satisfied as equality by the given points.

4 Separation heuristics for $P_{\mathcal{L}}$

In this section we consider some examples of separation heuristics for $P_{\mathcal{L}}$ than can be used in a Branch and Cut Method.

Given a solution x to the linear relaxation, by Lemma 2.4 we are looking for a dependent set in the matroid with values α and 1 in x, the 1's correspond to P and the others to T. Then look for an arc, independent from the cycle, with value $1-\alpha$ corresponding to q_0 and check if one of the corresponding inequalities is violated, if not, try to add elements to T, P and Q.

We define the submatroid M_α in the ground set $A = \{a \in E : x(a) = 1 \vee x(a) = \alpha \pm \epsilon\}$, $x(a)$ is the weight of a.

For each α present we could look in M_α either for small configurations, which we could enumerate or for a maximum base, this gives a cycles basis and we have a configuration for each element not in the basis or for a maximum 1-forest, that is a forest with one cycle.

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